Transport along stochatic flows in fluid dynamics

Valentin Resseguier, Etienne Mémin, Bertrand Chapron

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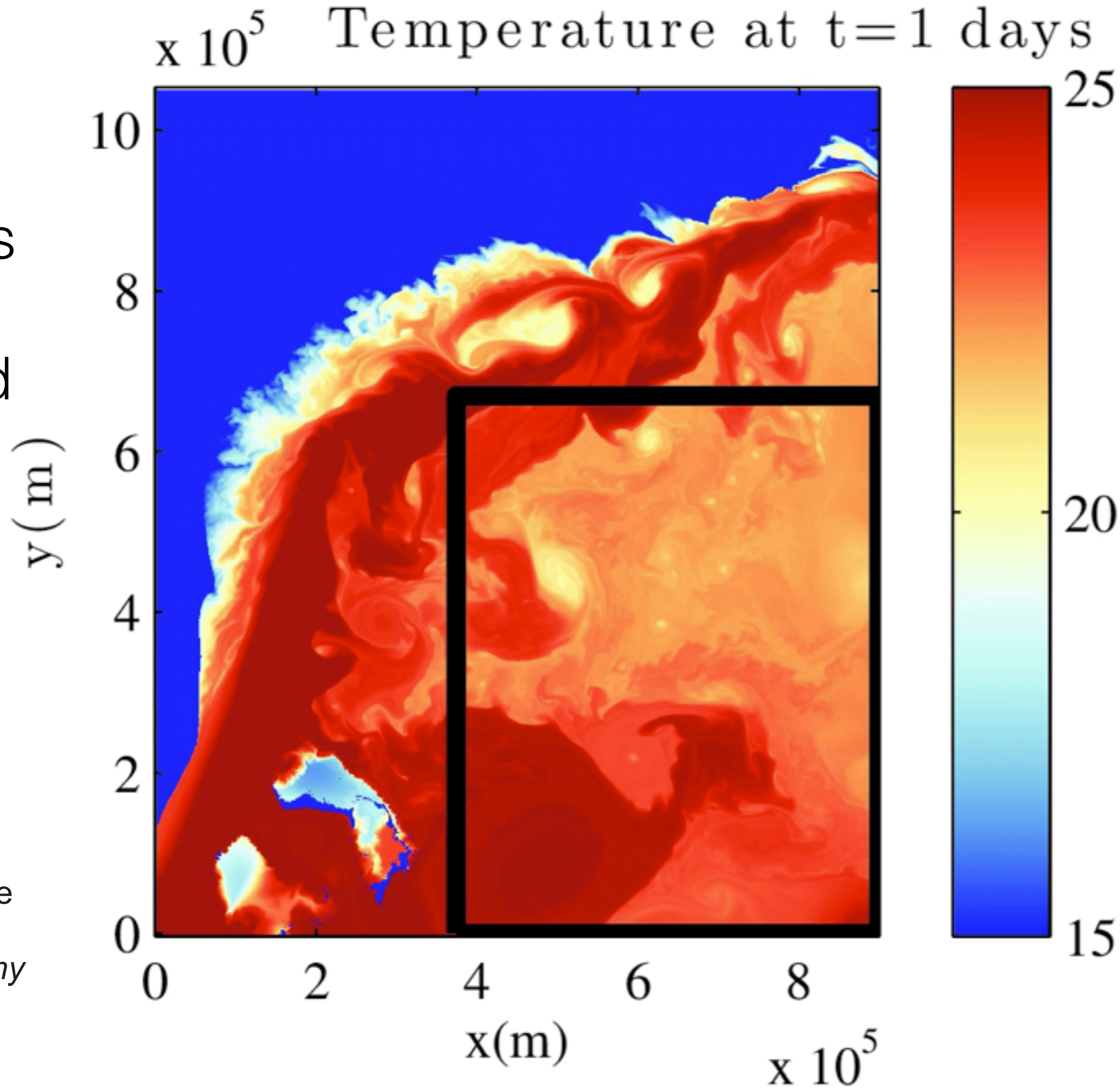
Fluids are very complex

with small vortices interacting with large vortices and currents



Gula, Jonathan, M. Jeroen Molemaker, and James C. McWilliams "Gulf Stream dynamics along the southeastern US seaboard." *Journal of Physical Oceanography* 45.3 (2015): 690-715.

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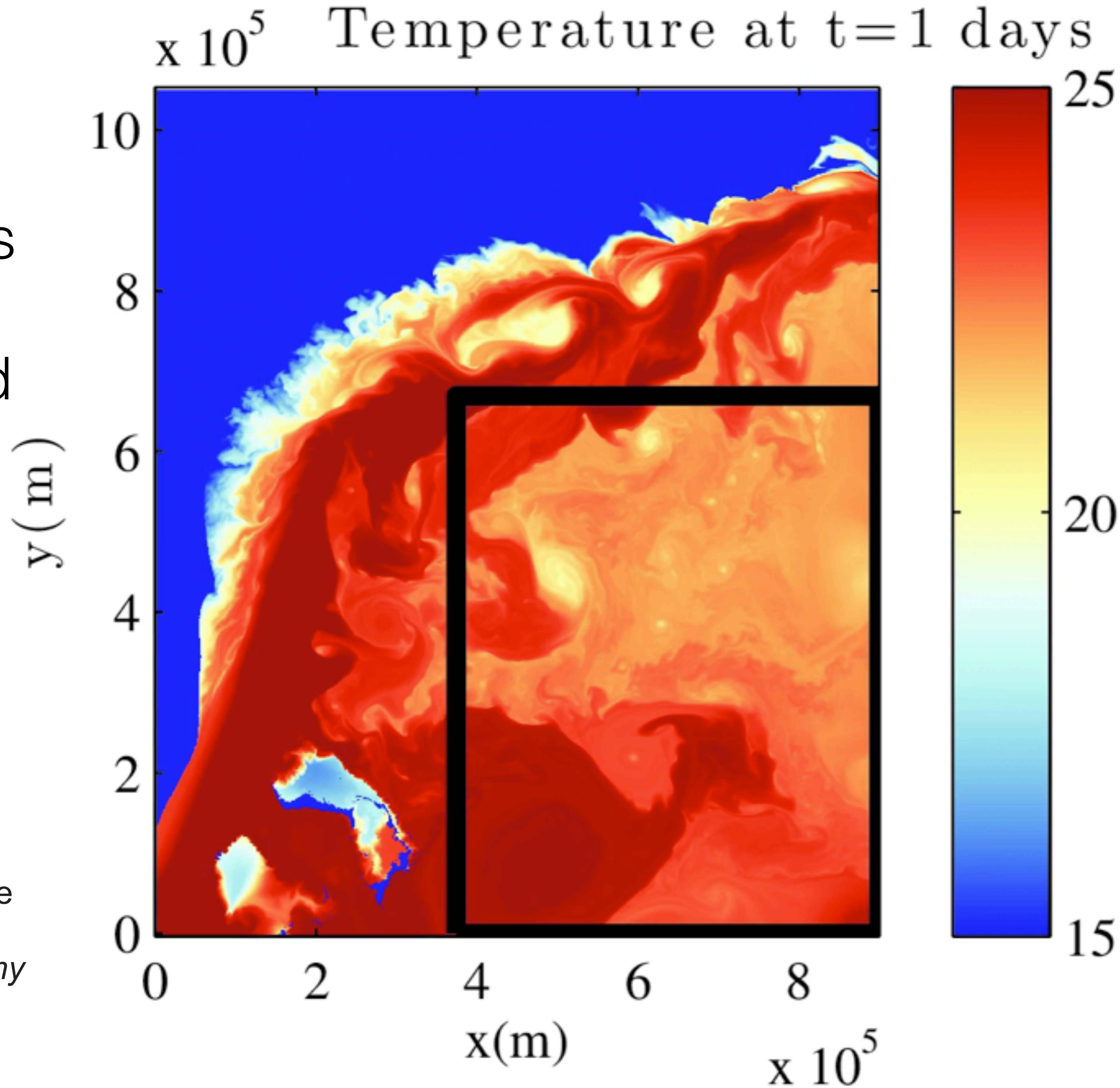

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Why a random fluid dynamics?

- Take into account unresolved processes (small scales)
- Physical justification of empirical models
- Predicting possible distinct scenarios, extreme-events, …
- Quantification of modeling errors for **data assimilation:**

  ensemble forecasts

Contents

- Randomized dynamics
- Simulation of the SQG under Moderate Uncertainty
- Stochastic reduced order model

Randomized dynamics

Advection of tracer Θ

- Stochastic flow:

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t$$

Advection of tracer Θ

- Stochastic flow:

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t$$

with infinite-dimensional Brownian motion

$$\boldsymbol{\sigma}(\bullet, t)\mathrm{d}\boldsymbol{B}_t \triangleq \int_{\Omega} \mathrm{d}\boldsymbol{z}\ \breve{\boldsymbol{\sigma}}(\bullet, \boldsymbol{z}, t)\mathrm{d}\boldsymbol{B}_t(\boldsymbol{z})$$

Advection of tracer Θ

- Stochastic flow:

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{w}(\boldsymbol{X}_t, t)\mathrm{d}t + \boldsymbol{\sigma}(\boldsymbol{X}_t, t)\mathrm{d}\boldsymbol{B}_t$$

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- A tracer is a function conserved along the flow:

$$D_t\Theta(t, \boldsymbol{X}_t) \triangleq \mathrm{d}\left[\Theta(t, \boldsymbol{X}_t)\right] = 0$$

Ito-Wentzel formula
(Kunita 1997)

If both $\boldsymbol{X}$ and $\boldsymbol{\Theta}$ are semimartingales (w.r.t. time) and $\boldsymbol{\Theta}$ is twice differentiable w.r.t. space

Then

$$\mathrm{d}\left[\Theta(t, \boldsymbol{X}(t,y))\right] = \mathrm{d}_t\Theta + (\nabla\Theta)^T\mathrm{d}\boldsymbol{X} + \frac{1}{2}tr(H_{\Theta}\mathrm{d} < \boldsymbol{X}, \boldsymbol{X}^T >)$$
$$+ \,\mathrm{d}_t < (\nabla\Theta)^T, \boldsymbol{X} >$$

Advection of tracer $\mathbf{\Theta}$

$$D_t \Theta = 0$$

Advection of tracer Θ

Advection of tracer $\boldsymbol{\Theta}$

$$\mathrm{d}_t\Theta + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta \mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)\mathrm{d}t$$

Advection of tracer $\boldsymbol{\Theta}$

Advection

$$\mathrm{d}_t\Theta + \boxed{\boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta \mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \boldsymbol{\nabla}\Theta} = \boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)\mathrm{d}t$$

Advection of tracer $\boldsymbol{\Theta}$

Advection

Diffusion

$$\mathrm{d}_t\Theta + \underbrace{\boldsymbol{w}^{\star}\cdot\boldsymbol{\nabla}\Theta\mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\Theta}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}_{\text{Diffusion}}\mathrm{d}t$$

Advection of tracer $\boldsymbol{\Theta}$

$$\mathrm{d}_t\Theta + \underbrace{\boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta \mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \boldsymbol{\nabla}\Theta}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla} \cdot \left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}_{\text{Diffusion}} \mathrm{d}t$$

$$\boldsymbol{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$$

$$\boldsymbol{w}^* = \boldsymbol{w} - \frac{1}{2}(\boldsymbol{\nabla} \cdot \boldsymbol{a})^T$$

Advection of tracer $\boldsymbol{\Theta}$

$$\mathrm{d}_t\Theta + \overbrace{\boldsymbol{w}^{\star}\cdot\boldsymbol{\nabla}\Theta\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\Theta}^{\text{Advection}} = \overbrace{\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}^{\text{Diffusion}}\mathrm{d}t$$

Drift correction

$$\boldsymbol{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$$

$$\boldsymbol{w}^{*} = \boldsymbol{w} - \frac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{a})^T$$

Advection of tracer $\boldsymbol{\Theta}$

Advection

Diffusion

$$\mathrm{d}_t\Theta + \boldsymbol{w}^\star \cdot \boldsymbol{\nabla}\Theta \mathrm{d}t + \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t \cdot \boldsymbol{\nabla}\Theta = \boldsymbol{\nabla}\cdot\left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)\mathrm{d}t$$

Drift correction

Multiplicative random forcing

$$\boldsymbol{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$$

$$\boldsymbol{w}^* = \boldsymbol{w} - \frac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{a})^T$$

Advection of tracer $\boldsymbol{\Theta}$

$$\mathrm{d}_t\Theta + \underbrace{\boldsymbol{w}^{\star}\cdot\boldsymbol{\nabla}\Theta\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\Theta}_{\text{Advection}} = \underbrace{\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}_{\text{Diffusion}}\mathrm{d}t$$

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Advection of tracer $\boldsymbol{\Theta}$

$$\mathrm{d}_t\Theta + \overbrace{\boldsymbol{w}^{\star}\cdot\boldsymbol{\nabla}\Theta\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{B}_t\cdot\boldsymbol{\nabla}\Theta}^{\text{Advection}} = \overbrace{\boldsymbol{\nabla}\cdot\left(\frac{1}{2}\boldsymbol{a}\boldsymbol{\nabla}\Theta\right)}^{\text{Diffusion}}\mathrm{d}t$$

Drift correction

Multiplicative random forcing

Balanced energy exchanges

$$\boldsymbol{a} = \boldsymbol{\sigma}\boldsymbol{\sigma}^T$$

$$\boldsymbol{w}^* = \boldsymbol{w} - \frac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{a})^T$$

$$\boldsymbol{\sigma} = ?$$

$$\boldsymbol{\sigma}(\bullet,t)\mathrm{d}\boldsymbol{B}_t \triangleq \int_\Omega \mathrm{d}\boldsymbol{z}\ \breve{\boldsymbol{\sigma}}(\bullet,\boldsymbol{z},t)\mathrm{d}\boldsymbol{B}_t(\boldsymbol{z})$$

- Parametric or non-parametric estimation
- Observations can be:
  - Small-scale eulerian velocity $\left(\boldsymbol{\sigma}(\boldsymbol{x}_i,t_j)\mathrm{d}\boldsymbol{B}_{t_j}\right)_{ij}$
  - Small-scale Lagrangian velocity $\left(\boldsymbol{\sigma}(\boldsymbol{X}_{t_j}(\boldsymbol{x}_i),t_j)\mathrm{d}\boldsymbol{B}_{t_j}\right)_{ij}$
  - **Tracer (solution of the SPDE)** $\left(\Theta(\boldsymbol{x}_i,t_j)\right)_{ij}$

- In the following simulations, very simple model (without estimations)

$$\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t = \int_\Omega \mathrm{d}\boldsymbol{z}\ \boldsymbol{\nabla}^\perp\breve{\psi}(\bullet-\boldsymbol{z})\mathrm{d}\boldsymbol{B}_t(\boldsymbol{z}) = \boldsymbol{\nabla}^\perp\breve{\psi} * \mathrm{d}\boldsymbol{B}_t$$

$$\widehat{\breve{\psi}}(\boldsymbol{k}) = A\ \mathbf{1}_{\{\kappa_1<\|\boldsymbol{k}\|<\kappa_2\}}\|\boldsymbol{k}\|^{-\alpha}$$

Simulation of the SQG under Moderate Uncertainty

SQG MU

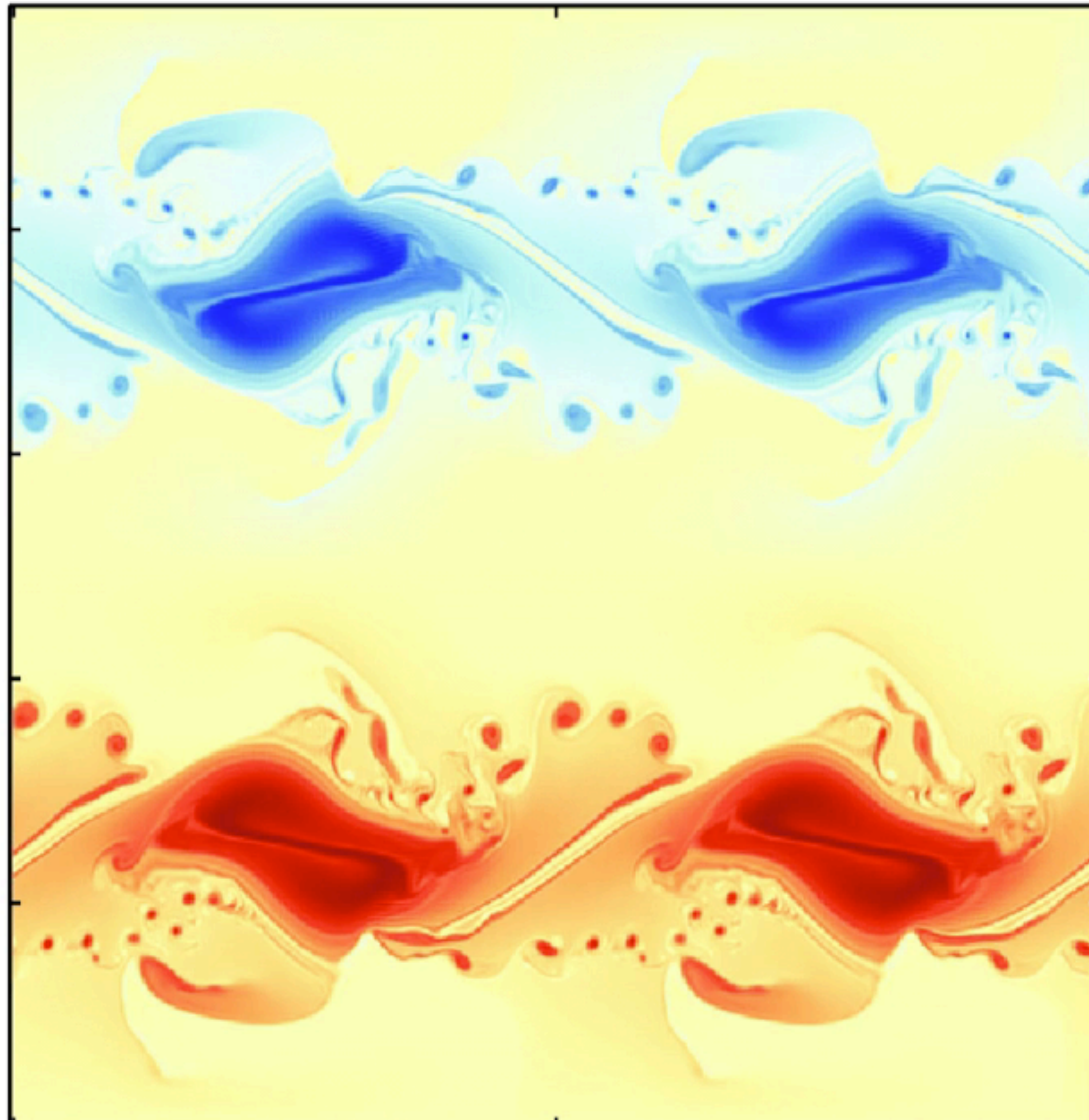
Code available online

$t = 17$ days

SQG model:

$$D_t \Theta = \nu \Delta^4 \Theta \mathrm{d}t$$

$$\boldsymbol{w} = \alpha \boldsymbol{\nabla}^{\perp}(-\Delta)^{-\frac{1}{2}}\Theta$$



Reference flow:

deterministic SQG

512 x 512

SQG model:

$$D_t\Theta = \nu\Delta^4\Theta \mathrm{d}t$$

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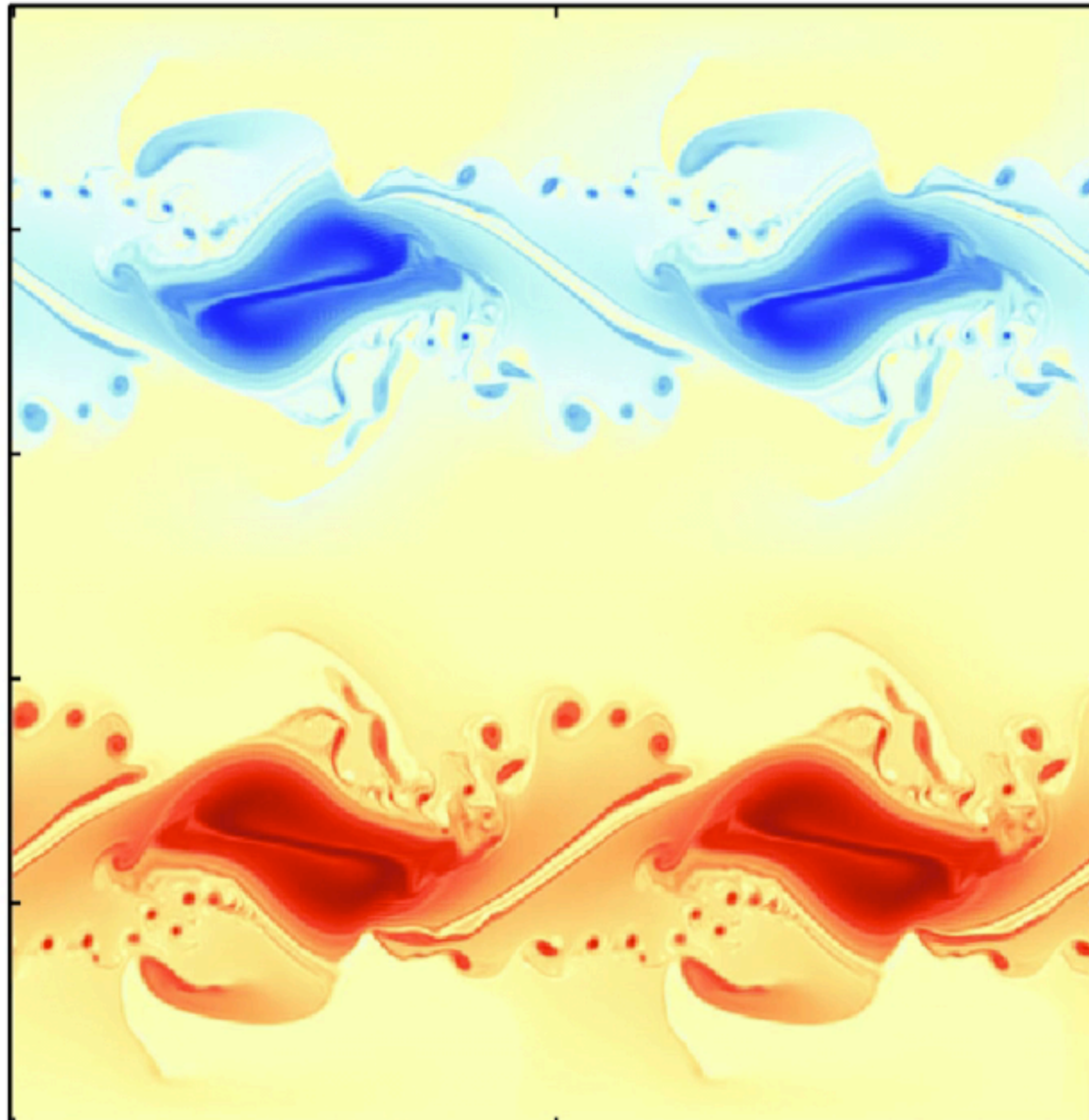
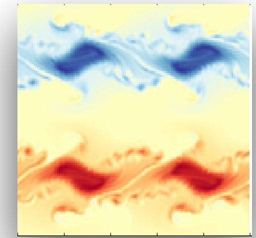
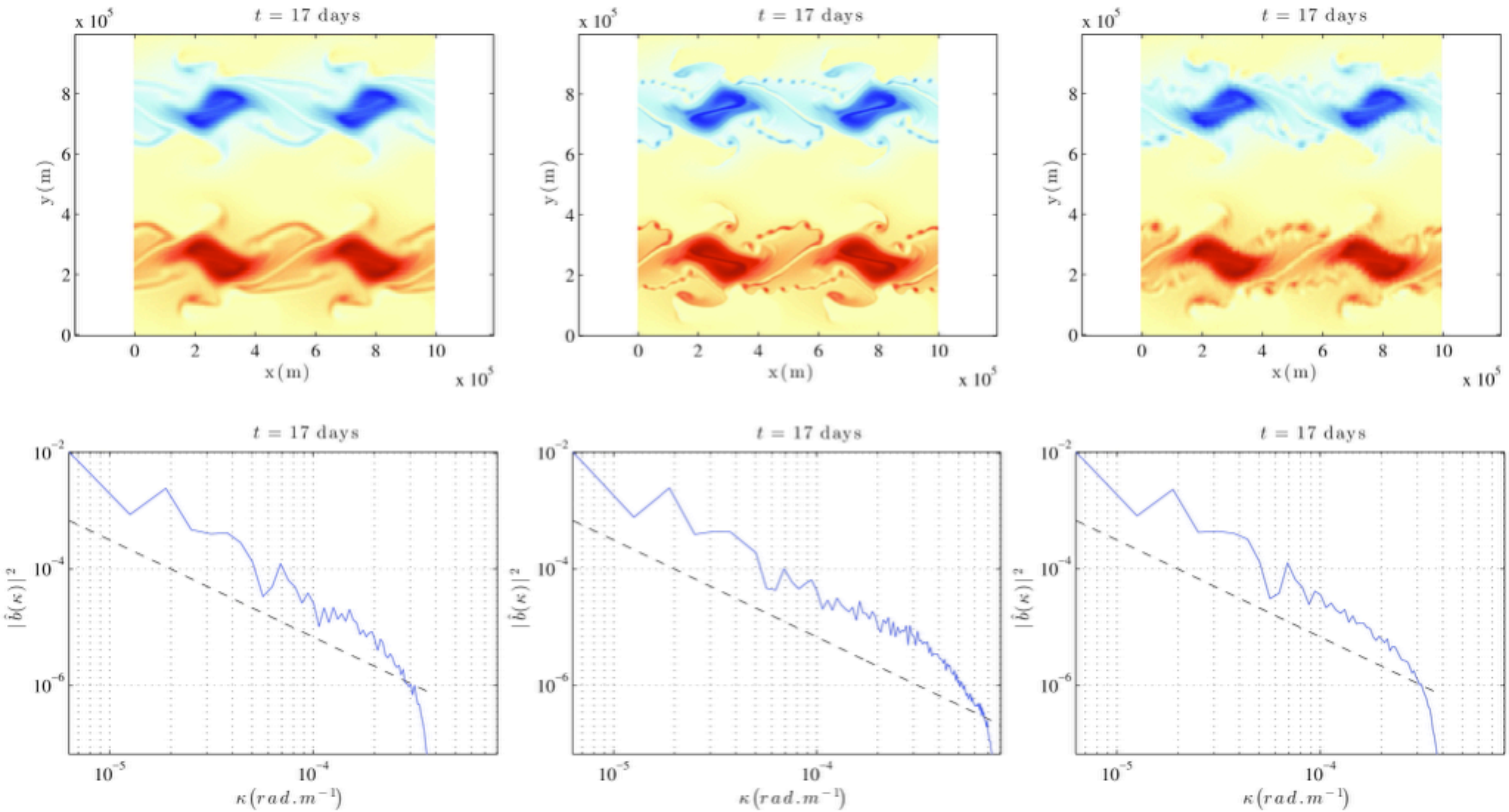
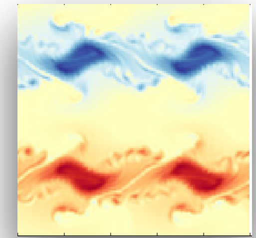




One realization



Deterministic 128x128 **Deterministic 512x512** **Stochastic 128x128**



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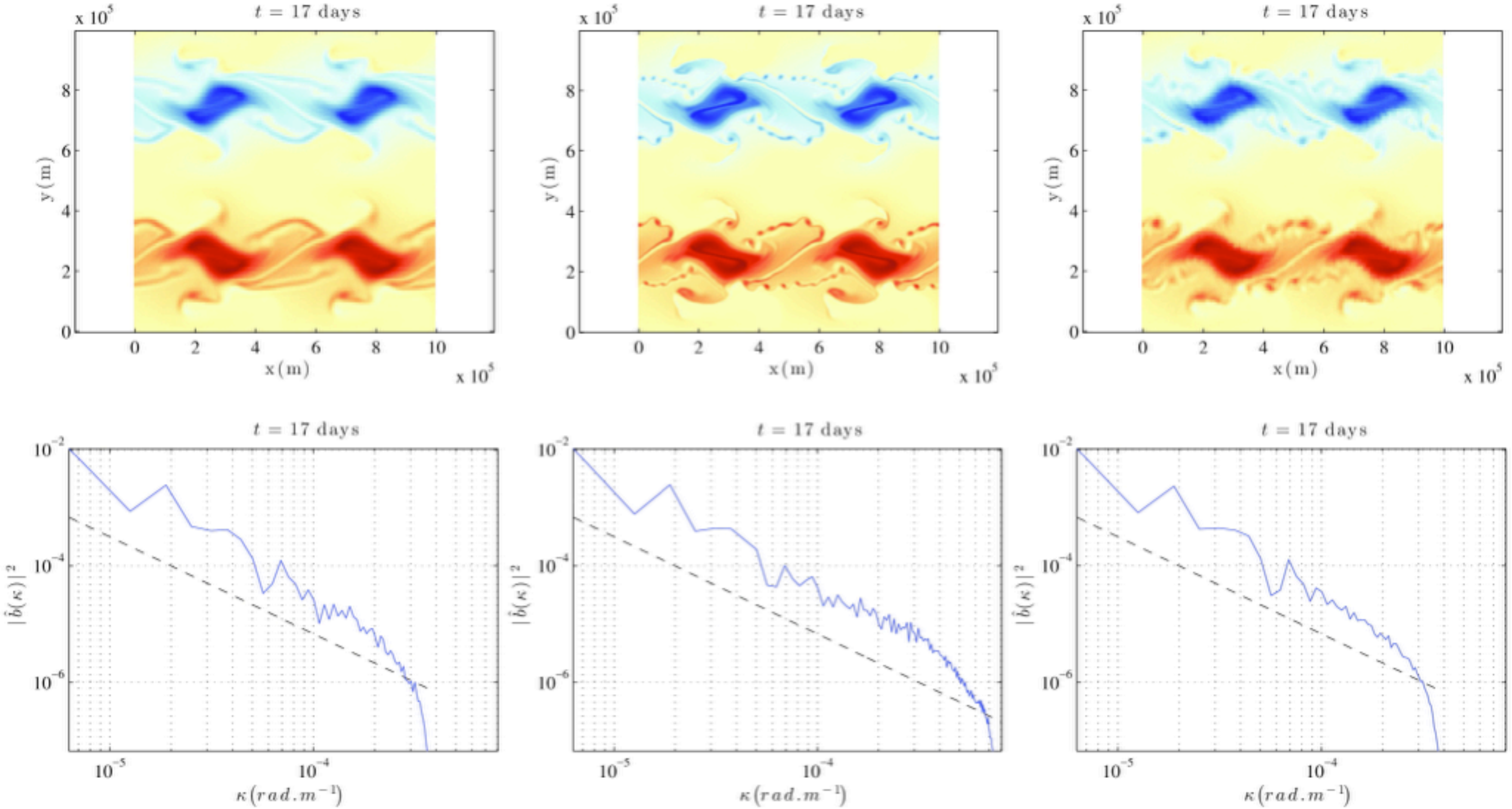
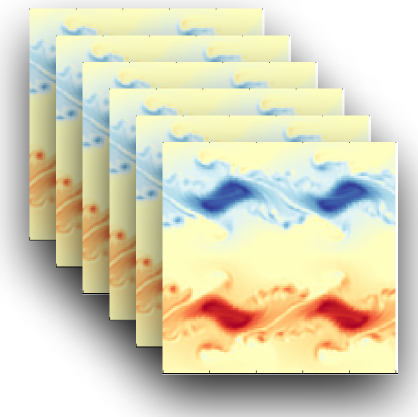
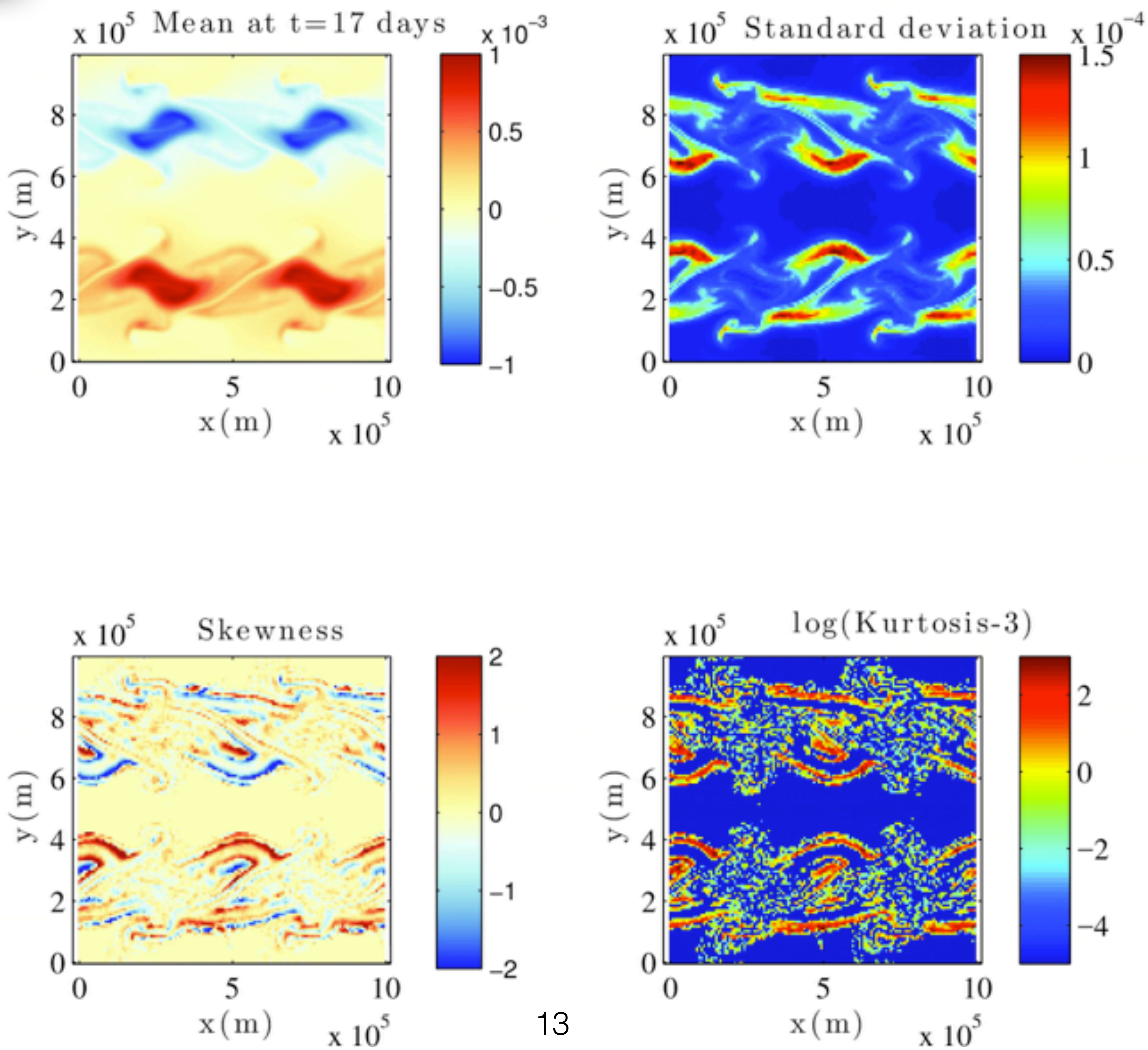


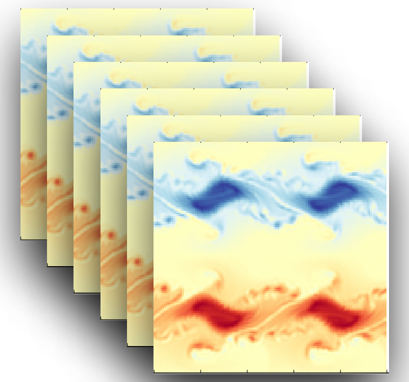
Deterministic 128x128 **Deterministic 512x512** **Stochastic 128x128**



Ensemble



Mean at t=17 days
x 10^5
x 10^-3
y (m)
x (m)
x 10^5
Standard deviation
x 10^-4
Skewness
log(Kurtosis-3)



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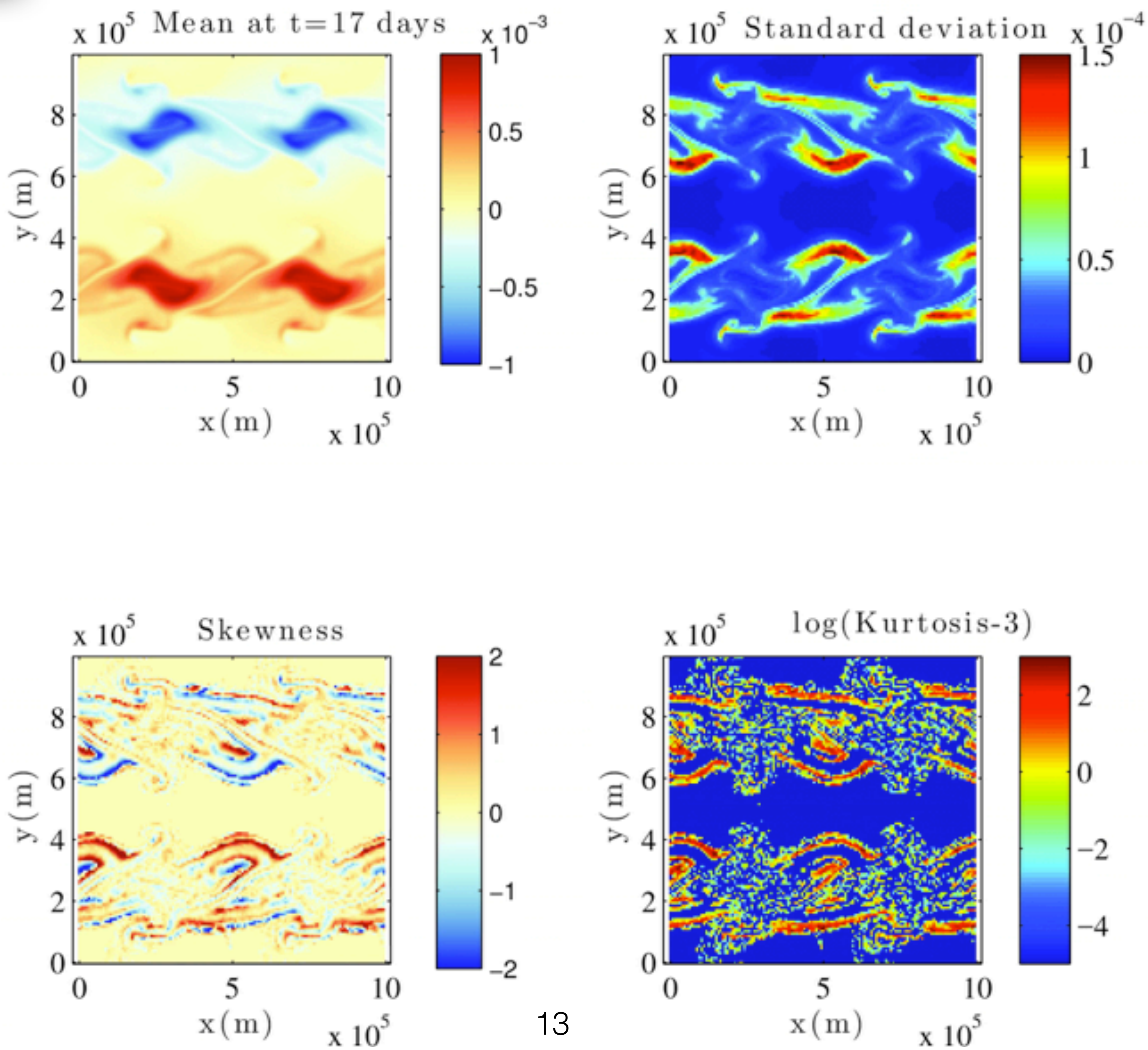

Mean at t=17 days
Standard deviation
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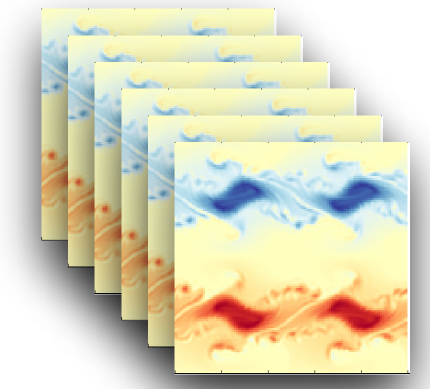
x 10^5
x 10^-3
x 10^-4
y (m)
x (m)



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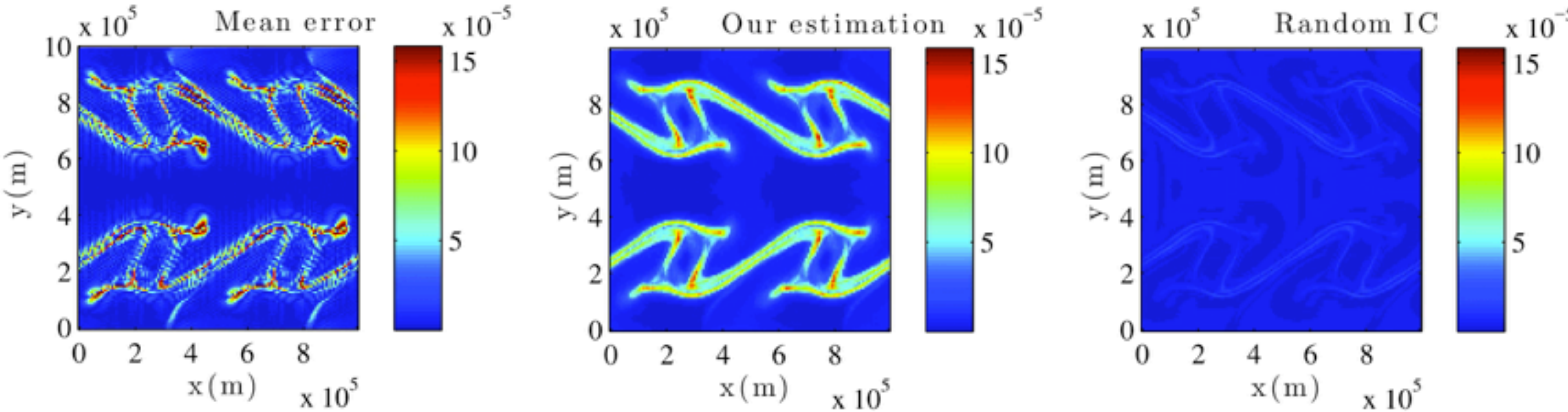
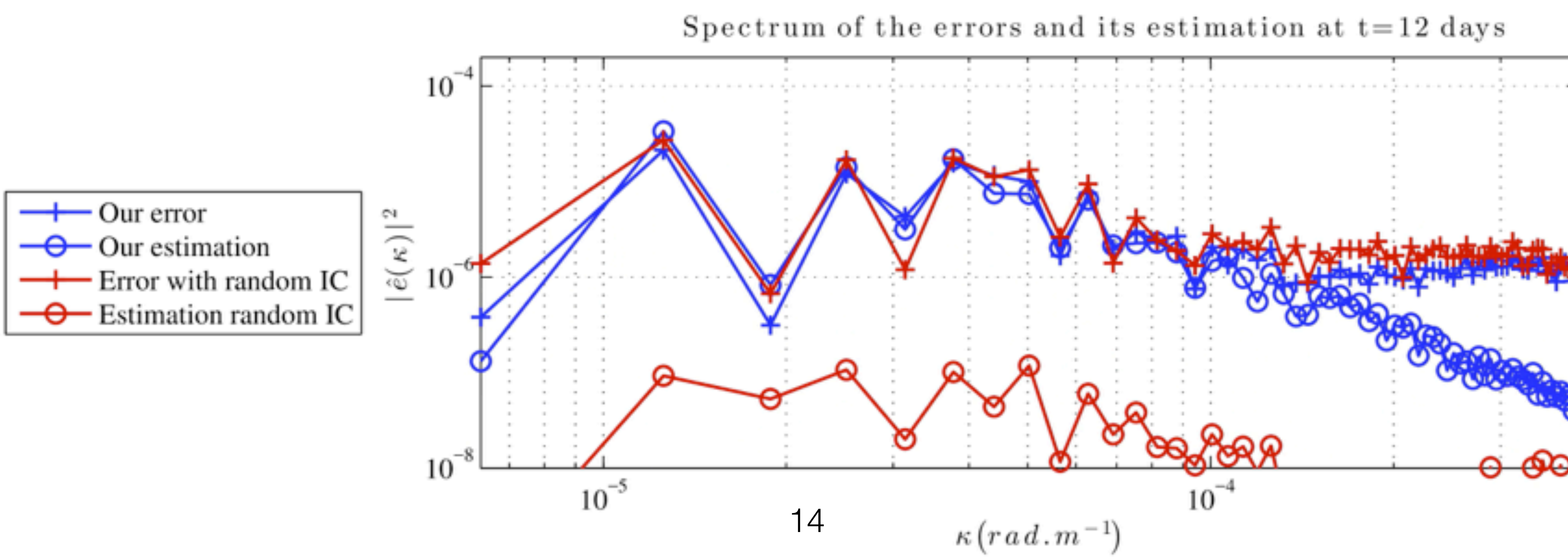
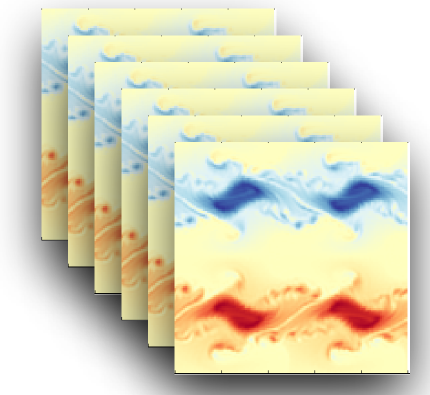


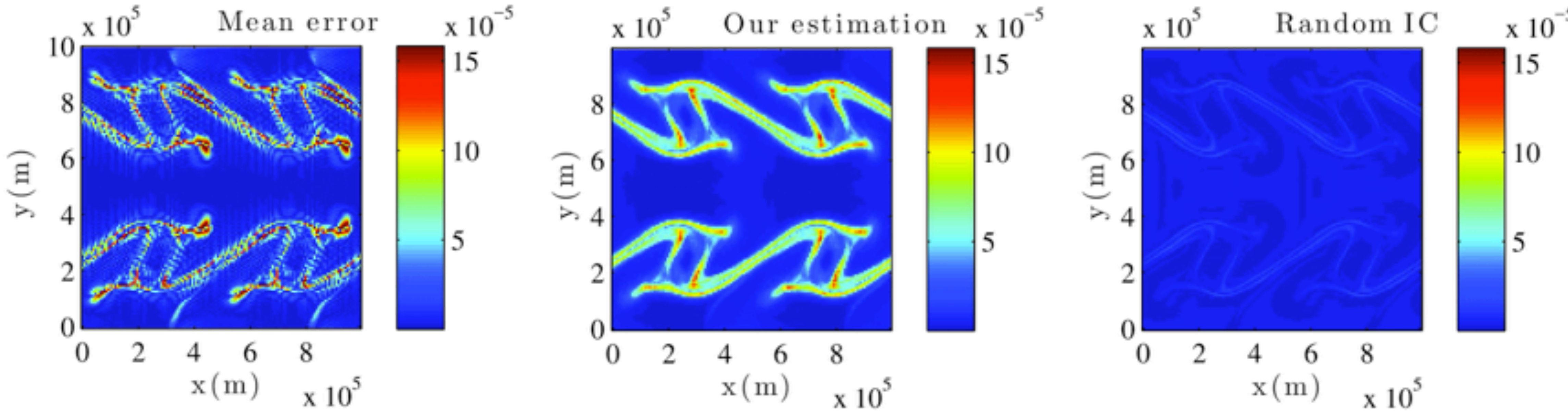
Mean error
Our estimation
Random IC
x 10^5
x 10^-5
y (m)
x (m)

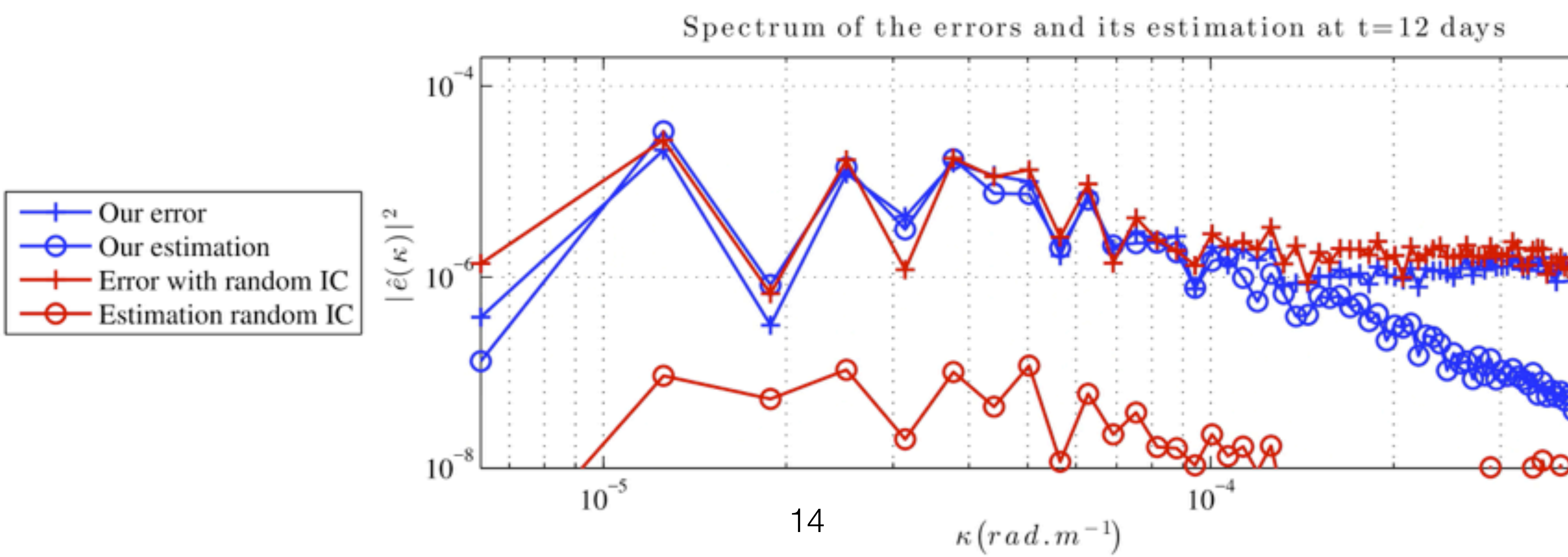


Spectrum of the errors and its estimation at t=12 days
Our error
Our estimation
Error with random IC
Estimation random IC
$|\hat{e}(\kappa)|^2$
$\kappa\,(rad.m^{-1})$



Ensemble


Mean error
Our estimation
Random IC
x 10^5
x 10^-5
y (m)
x (m)
0
2
4
6
8
10
5
15


Spectrum of the errors and its estimation at t=12 days
Our error
Our estimation
Error with random IC
Estimation random IC
$|\hat{e}(\kappa)|^2$
$\kappa\,(rad.m^{-1})$
10^{-4}
10^{-6}
10^{-8}
10^{-5}
10^{-4}

Random reduced order model from stochastic Navier-Stokes equation

Why a reduced order model?

- Very fast simulation of very complex system (e.g. for industrial application)

What is a reduced order model?

- Physical model (PDE) simplified using observations

PCA on data to reduce the state-space dimension

$$\boldsymbol{v}(\boldsymbol{x},t) = \overline{\boldsymbol{v}}(\boldsymbol{x}) + \sum_{i=0}^{n} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x}) + \sum_{i=n+1}^{N} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})$$

PCA on data to reduce the state-space dimension

$$\boldsymbol{v}(\boldsymbol{x},t) = \overbrace{\boxed{\overline{\boldsymbol{v}}(\boldsymbol{x}) + \sum_{i=0}^{n} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})}}^{\text{Resolved modes}} + \sum_{i=n+1}^{N} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})$$

$\boldsymbol{w}$

PCA on data to reduce the state-space dimension

Resolved modes

Unresolved modes

$$\boldsymbol{v}(\boldsymbol{x},t) = \overline{\boldsymbol{v}}(\boldsymbol{x}) + \sum_{i=0}^{n} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x}) + \sum_{i=n+1}^{N} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})$$

$\boldsymbol{w}$

$$\approx \frac{1}{\Delta t}\,\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t$$

PCA on data to reduce the state-space dimension

Resolved modes **Unresolved modes**

$$\boldsymbol{v}(\boldsymbol{x},t) = \boxed{\overline{\boldsymbol{v}}(\boldsymbol{x}) + \sum_{i=0}^{n} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})} + \boxed{\sum_{i=n+1}^{N} b_i(t)\boldsymbol{\phi}_i(\boldsymbol{x})}$$

$\boldsymbol{w}$

	Initial space	Reduced subspace
Solution coordinates	$(w_q(\boldsymbol{x}_i,t))_{qi}$	$(b_i(t))_i$
Dimension	d = 2 or 3 x M ~10^7	n~10

$$\approx \frac{1}{\Delta t}\boldsymbol{\sigma}\,\mathrm{d}\boldsymbol{B}_t$$

Assumptions of finite variations for $\boldsymbol{w}$ and
Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \boldsymbol{\phi}_i \cdot \text{(stochastic Navier-Stokes)}$$

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t$$

Assumptions of finite variations for $\boldsymbol{w}$ and Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \boldsymbol{\phi}_i \cdot \text{(stochastic Navier-Stokes)}$$

$$\mathrm{d}b_i = \boxed{F_i(\boldsymbol{b})}\mathrm{d}t$$

2nd order polynomial:
coefficients given by physics,

$(\boldsymbol{\phi}_j)_j$ and $\boldsymbol{a}(\boldsymbol{x},\boldsymbol{x},t) = \dfrac{1}{t} < (\boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{B})_{obs}, (\boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{B})_{obs}^T$

Assumptions of finite variations for $\boldsymbol{w}$ and Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \boldsymbol{\phi}_i \cdot \text{(stochastic Navier-Stokes)}$$

$$\mathrm{d}b_i = \boxed{F_i(\boldsymbol{b})}\mathrm{d}t$$

Model and estimation
(inspired by Genon-Catalot, Laredo, Picard 1992)

$$\boldsymbol{a}(\boldsymbol{x},\boldsymbol{x},t) = \boldsymbol{z}_0(\boldsymbol{x}) + \sum_{i=0}^{n} b_i(t)\boldsymbol{z}_i(\boldsymbol{x})$$

$$\boldsymbol{z}_i(\boldsymbol{x}) = \frac{1}{t}\int_0^t b_i(t)\mathrm{d} < \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{B}, \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{B} >$$

2nd order polynomial:
coefficients given by physics,

$(\boldsymbol{\phi}_j)_j$ and $\boldsymbol{a}(\boldsymbol{x},\boldsymbol{x},t) = \dfrac{1}{t} < (\boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{B})_{obs}, (\boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{B})^T_{obs}$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \boldsymbol{\phi}_i \cdot \text{(stochastic Navier-Stokes)}$$

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + (\boldsymbol{\alpha}_{\bullet i}\mathrm{d}\boldsymbol{B}_t)^T \boldsymbol{b} + (\boldsymbol{\theta}_i \mathrm{d}\boldsymbol{B}_t)$$

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$$\int_{\Omega} \boldsymbol{\phi}_i \cdot \text{(stochastic Navier-Stokes)}$$

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + (\overbrace{\boxed{\boldsymbol{\alpha}_{\bullet i}}}^{\text{n x M}}\overbrace{\boxed{\mathrm{d}\boldsymbol{B}_t}}^{\text{M x 1}})^T \overbrace{\boxed{\boldsymbol{b}}}^{\text{n x 1}} + (\overbrace{\boxed{\boldsymbol{\theta}_i}}^{\text{1 x M}}\overbrace{\boxed{\mathrm{d}\boldsymbol{B}_t}}^{\text{M x 1}})$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \boldsymbol{\phi}_i \cdot \text{(stochastic Navier-Stokes)}$$

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + \underbrace{\Big(\overset{\text{n x M}}{\boldsymbol{\alpha}_{\bullet i}}\,\overset{\text{M x 1}}{\mathrm{d}\boldsymbol{B}_t}\Big)^T \overset{\text{n x 1}}{\boldsymbol{b}}}_{\text{multiplicative noise}} + \underbrace{\Big(\overset{\text{1 x M}}{\boldsymbol{\theta}_i}\,\overset{\text{M x 1}}{\mathrm{d}\boldsymbol{B}_t}\Big)}_{\text{additive noise}}$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_\Omega \boldsymbol{\phi}_i \cdot \text{(stochastic Navier-Stokes)}$$

$$\mathrm{d}b_i = \boxed{F_i(\boldsymbol{b})}\mathrm{d}t + \underbrace{\boxed{(\underbrace{\boxed{\boldsymbol{\alpha}_{\bullet i}}}_{n \times M} \underbrace{\boxed{\mathrm{d}\boldsymbol{B}_t}}_{M \times 1})^T \underbrace{\boxed{\boldsymbol{b}}}_{n \times 1}}}_{\text{multiplicative noise}} + \underbrace{\boxed{(\underbrace{\boxed{\boldsymbol{\theta}_i}}_{1 \times M} \underbrace{\boxed{\mathrm{d}\boldsymbol{B}_t}}_{M \times 1})}}_{\text{additive noise}}$$

2nd order polynomial:
coefficients given by physics,

$\left(\boldsymbol{\phi}_j\right)_j$ and $\boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \dfrac{1}{t} < (\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{B})_{obs}, (\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{B})_{obs}^T >_t$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot \text{(stochastic Navier-Stokes)}$$

n x M M x 1 n x 1 1 x M M x 1

$$\mathrm{d}b_i = \boxed{F_i(\boldsymbol{b})}\mathrm{d}t + \boxed{(\boxed{\boldsymbol{\alpha}_{\bullet i}}\boxed{\mathrm{d}\boldsymbol{B}_t})^T\boxed{\boldsymbol{b}}} + \boxed{(\boxed{\boldsymbol{\theta}_i}\boxed{\mathrm{d}\boldsymbol{B}_t})}$$

multiplicative noise additive noise

Correlations to estimate

2nd order polynomial:
coefficients given by physics,

$$\left(\boldsymbol{\phi}_j\right)_j \quad \text{and} \quad \boldsymbol{a}(\boldsymbol{x},\boldsymbol{x}) = \frac{1}{t} < (\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{B})_{obs}, (\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{B})_{obs}^T >_t$$

Estimate of the correlation $\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}_{qj}^T$

SDE

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + \boxed{(\boldsymbol{\alpha}_{\bullet i}\mathrm{d}\boldsymbol{B}_t)^T\boldsymbol{b}} + (\boldsymbol{\theta}_i\mathrm{d}\boldsymbol{B}_t)$$

multiplicative noise

Classical estimate

$$\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}_{qj}^T = \frac{1}{t}\langle \boldsymbol{\alpha}_{pi}\boldsymbol{B}, \boldsymbol{\alpha}_{qj}\boldsymbol{B}\rangle_t$$

Problem:

- $$(\underbrace{\boxed{\boldsymbol{\alpha}_{pi}}}_{1 \times M}\underbrace{\boxed{\mathrm{d}\boldsymbol{B}_t}}_{M \times 1})_{obs} = \int_\Omega \underbrace{\boxed{\boldsymbol{G}_{pi}}}_{\text{Known but complex linear operator}}[(\underbrace{\boxed{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}}_{d \times M \text{ (as a function of space)}})_{obs}]$$

  Too complex to be computed for each time step

-

Estimate of the correlation $\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}_{qj}^T$

SDE

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + \boxed{(\boldsymbol{\alpha}_{\bullet i}\mathrm{d}\boldsymbol{B}_t)^T\boldsymbol{b}} + (\boldsymbol{\theta}_i\mathrm{d}\boldsymbol{B}_t)$$

multiplicative noise

Classical estimate

$$\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}_{qj}^T = \frac{1}{t}\langle\boldsymbol{\alpha}_{pi}\boldsymbol{B}, \boldsymbol{\alpha}_{qj}\boldsymbol{B}\rangle_t$$

Problem:

- $$(\underbrace{\boxed{\boldsymbol{\alpha}_{pi}}}_{1 \times M}\underbrace{\boxed{\mathrm{d}\boldsymbol{B}_t}}_{M \times 1})_{obs} = \int_\Omega \underbrace{\boxed{\boldsymbol{G}_{pi}}}_{\text{Known but complex linear operator}}[(\underbrace{\boxed{\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_t}}_{\text{d x M (as a function of space)}})_{obs}]$$ ➔ Too complex to be computed for each time step

- $$\underbrace{\boxed{\boldsymbol{a}(\boldsymbol{x},\boldsymbol{y})}}_{\text{d x d x M x M}} = \frac{1}{t} < (\boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{B})_{obs}, (\boldsymbol{\sigma}(\boldsymbol{y})\boldsymbol{B})_{obs}^T >_t$$ ➔ Cannot be even memorized

Estimate of the correlation $\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}^T_{qj}$

SDE

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + \boxed{(\boldsymbol{\alpha}_{\bullet i}\mathrm{d}\boldsymbol{B}_t)^T\boldsymbol{b}} + (\boldsymbol{\theta}_i\mathrm{d}\boldsymbol{B}_t)$$

multiplicative noise

Solution:

- $$\frac{1}{t}\left\langle b_i, \int_0^t b_p\,(\boldsymbol{\alpha}_{qj}\mathrm{d}\boldsymbol{B}_t)\right\rangle_t = \sum_k \boldsymbol{\alpha}_{ki}\boldsymbol{\alpha}^T_{qj}\underbrace{\frac{1}{t}\int_0^t b_k b_p}_{=\lambda_p\delta_{kp}} + \boldsymbol{\theta}_i\boldsymbol{\alpha}^T_{qj}\underbrace{\frac{1}{t}\int_0^t b_p}_{=0}$$

  (from the PCA)

  $$= \lambda_p\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}^T_{qj}$$

-

Estimate of the correlation $\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}_{qj}^T$

SDE

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + \boxed{(\boldsymbol{\alpha}_{\bullet i}\mathrm{d}\boldsymbol{B}_t)^T\boldsymbol{b}} + (\boldsymbol{\theta}_i\mathrm{d}\boldsymbol{B}_t)$$

multiplicative noise

Solution:

- $$\frac{1}{t}\left\langle b_i, \int_0^t b_p\,(\boldsymbol{\alpha}_{qj}\mathrm{d}\boldsymbol{B}_t)\right\rangle_t = \sum_k \boldsymbol{\alpha}_{ki}\boldsymbol{\alpha}_{qj}^T \underbrace{\frac{1}{t}\int_0^t b_k b_p}_{=\lambda_p\delta_{kp}} + \boldsymbol{\theta}_i\boldsymbol{\alpha}_{qj}^T \underbrace{\frac{1}{t}\int_0^t b_p}_{=0}$$

  (from the PCA)

  $$= \lambda_p \boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}_{qj}^T$$

- $$\frac{1}{t}\left\langle b_i, \int_0^t b_p\,(\boldsymbol{\alpha}_{qj}\mathrm{d}\boldsymbol{B}_{t'})\right\rangle_t = \int_\Omega \boldsymbol{G}_{qj}\left[\frac{1}{t}\left\langle (b_i)_{obs}, \int_0^t (b_p\;\boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_{t'})_{obs}\right\rangle_t\right]$$

By linearity of G_{qj}

Estimate of the correlation $\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}^T_{qj}$

SDE

$$\mathrm{d}b_i = F_i(\boldsymbol{b})\mathrm{d}t + \boxed{(\boldsymbol{\alpha}_{\bullet i}\mathrm{d}\boldsymbol{B}_t)^T\boldsymbol{b}} + (\boldsymbol{\theta}_i\mathrm{d}\boldsymbol{B}_t)$$

multiplicative noise

Solution:

(from the PCA)

- $$\frac{1}{t}\left\langle b_i, \int_0^t b_p\,(\boldsymbol{\alpha}_{qj}\mathrm{d}\boldsymbol{B}_t)\right\rangle_t = \sum_k \boldsymbol{\alpha}_{ki}\boldsymbol{\alpha}^T_{qj}\underbrace{\frac{1}{t}\int_0^t b_k b_p}_{=\lambda_p\delta_{kp}} + \boldsymbol{\theta}_i\boldsymbol{\alpha}^T_{qj}\underbrace{\frac{1}{t}\int_0^t b_p}_{=0}$$
  $$= \lambda_p\boldsymbol{\alpha}_{pi}\boldsymbol{\alpha}^T_{qj}$$

d x M

- $$\frac{1}{t}\left\langle b_i, \int_0^t b_p\,(\boldsymbol{\alpha}_{qj}\mathrm{d}\boldsymbol{B}_{t'})\right\rangle_t = \int_\Omega \boldsymbol{G}_{qj}\left[\boxed{\frac{1}{t}\left\langle (b_i)_{obs}, \int_0^t (b_p\ \boldsymbol{\sigma}\mathrm{d}\boldsymbol{B}_{t'})_{obs}\right\rangle_t}\right]$$

Independent of time t !

By linearity of G_{qj}

Conclusion

Conclusion

- Random transport applicable to any fluid dynamics models
- Better small scales
- Estimate position and amplitude of errors, extreme events, likely scenarios
- Possible applications for your previous and future estimation methods (e.g. MLE with $(\Theta(\boldsymbol{x}_i, t_j))_{ij}$)

Thank you for your attention

Code SQG MU:
link from Fluminance website - V. Resseguier